Journal of Statistical Physics, Vol. 127, No. 6, June 2007 (© 2007)
DOI: 10.1007/s10955-007-9299-8

Hybrid Quasicrystals, Transport and Localization in Products of Minimal Sets

Túlio O. Carvalho[1] and César R. de Oliveira[2,3]

Received July 31, 2006; accepted February 7, 2007
Published Online: May 2, 2007

We consider convex combinations of finite-valued almost periodic sequences (mainly substitution sequences) and put them as potentials of one-dimensional tight-binding models. We prove that these sequences are almost periodic. We call such combinations *hybrid quasicrystals* and these studies are related to the minimality, under the shift on both coordinates, of the product space of the respective (minimal) hulls. We observe a rich variety of behaviors on the quantum dynamical transport ranging from localization to transport.

KEY WORDS: almost periodic sequences, hybrid quasicrystals, transport, localization.

1. INTRODUCTION

The study of transport in one-dimensional aperiodic lattices may be modeled by the nearest-neighbors tight-binding Hamiltonian (Schrödinger operator) in $l^2(\mathbb{Z})$

$$(H\psi)_n = \psi_{n+1} + \psi_{n-1} + \lambda V_n \psi_n, \tag{1}$$

with $\lambda > 0$ and potentials $V = (V_n)_{n\in\mathbb{Z}}$ generated by aperiodic sequences. In many circumstances the potentials are real-valued functions of sequences on a finite set $\mathcal{A}$, called *alphabet*; these are models of one-dimensional quasicrystals.[(2)]

[1] Departamento de Matemática–UEL, CP 6001, Londrina, PR 86051-970, Brazil.
[2] Departamento de Matemática–UFSCar, São Carlos, SP 13560-970, Brazil.
[3] CRdO thanks the partial support by CNPq.

Quantum interferences may lead to localization of the solutions of the corresponding Schrödinger equation

$$i \frac{\partial}{\partial t} \psi(t) = H\psi(t),$$

as in case of (random) Bernoulli potentials,[12] but also to ballistic motion, mainly related to periodic potentials.

Among the characterizations of localization and transport we single out the second moment of the position operator

$$m_2(T) := \sum_{n=-\infty}^{\infty} |n - n_0|^2 |\psi_n(T)|^2, \tag{2}$$

usually with initial condition concentrated on a single site n_0. For a large class of potentials the moment $m_2(T) \leq CT^2$ (at least for $T > 1$) and if $m_2(T) \approx CT^2$ holds we have the definition of *ballistic motion*. Localization is characterized by a bounded function $m_2(T) \leq C$, $\forall T$; lack of localization is usually referred to as *delocalization* or *transport*. Half the way between these extremes are the *anomalous transport*, that is,

$$m_2(T) \approx CT^\beta \quad \text{with} \quad 0 < \beta < 2,$$

which are usually accompanied by singular continuous spectrum of the operator H. Important examples of such anomalous behavior are the above cited models of quasicrystals, among which the most prominent are the (primitive) substitution sequences,[2,18] for instance, Fibonacci, Thue-Morse and Period Doubling sequences. The Schrödinger operators whose potentials are generated by these sequences have singular continuous spectrum of zero Lebesgue measure (see ref. [8] and references therein).

A widespread spectral point of view makes the association of singular continuous spectrum with anomalous transport, absolutely continuous to ballistic motion and point spectrum of the Schrödinger operator with localization, even though there are known exceptions, namely of operators with purely point spectra showing transport. Even rank one perturbation (a very localized one) can exchange point and singular continuous spectra,[20] and the latter surely implies transport (any continuous spectrum does, as a consequence of RAGE theorem). What about unlocalized perturbations, i.e., those spread over the whole lattice? Certainly this becomes a too huge class of problems to be reasonably dealt with.

However, there is a special type of such perturbations we think it is worth considering and may be of some (experimental) relevance in the near future. A particular model of quasicrystal, as a substitution sequence, is an almost periodic sequence that grows up from a seed (i.e., an initial condition) and a specific "growing rule." If one has control of the growing technique, one could grow a quasicrystal in one direction following one such rule, and in a perpendicular direction following another rule. This hybridization creates a potential which is a linear (convex) combinations of the original ones. This type of long range perturbations of the potentials can also be considered from the theoretical point of view, the sequence spaces are two-dimensional and have been considered before.[21]

The potentials we shall consider are constructed as follows. Given two *parent* potentials $v = (v_n)_{n\in\mathbb{Z}}$, $u = (u_n)_{n\in\mathbb{Z}}$ and $0 \le \kappa \le 1$, the *hybrid potential* is

$$\mathcal{I}_\kappa(v, u) := \kappa v + (1-\kappa)u = (\kappa v_n + (1-\kappa)u_n)_{n\in\mathbb{Z}}\,.$$

Experience with random potentials indicates that if one of them is random then this characteristic will prevail with respect to localization. If both potentials are periodic, then the resulting one will also be periodic with period given by their least common multiple. So, in these extreme cases again, localization and ballistic motion, respectively, are persistent. Note that the number of values a hybrid potential assumes is in general larger than the number of values of each of its components; e.g., if both v, u take values in $\{0, 1\}$, then $\mathcal{I}_\kappa(v, u)$ will generally assume all values in $\{0, \kappa, (1-\kappa), 1\}$.

This work is an initial study on this proposal, and we will limit ourselves to almost periodic potentials taking a finite number of values (notably, substitution sequences). We present theoretical results on minimality, and data for the moment $m_2(T)$ from numerical time evolution simulations.

Section 2 review briefly some aspects of finitely valued sequences. In Sec. 3 we address the question about minimality of the product of minimal sets, giving a sufficient condition for it. In Sec. 4 we report some outcomes of numerical simulations of the moment $m_2(T)$ for the hybrid potentials, closing in the final section with our concluding remarks.

2. SUMMARY ON SEQUENCES AND SUBSTITUTIONS

We denote by $\mathcal{A}^*$ (resp. $\mathcal{A}^{\mathbb{Z}}$) the sets of finite (resp. bi-infinite) words with letters in the finite set $\mathcal{A}$ (called *alphabet*), which can be considered a subset of the real numbers. The metric on $\mathcal{A}^{\mathbb{Z}}$ is

$$d(a, b) = \begin{cases} 0, & \text{if } \forall n \in \mathbb{Z}, a_n = b_n \\ \frac{1}{2^n}, & \text{where } n = \min\{|j| : a_j \ne b_j\}. \end{cases}$$

A dynamics on this set is the (left) shift $(\sigma(v))_n = v_{n+1}$. Recall that a sequence $v \in \mathcal{A}^{\mathbb{Z}}$ is almost periodic iff its hull (the bar indicates the closure of the set)

$$\Omega(v) := \overline{\{\sigma^j(v) : j \in \mathbb{Z}\}}$$

is minimal, that is, the hull of any sequence in $\Omega(v)$ coincides with $\Omega(v)$. The set $\mathcal{O}(v) = \{\sigma^j(v) : j \in \mathbb{Z}\}$ is the orbit of v. By Tychonov theorem $\mathcal{A}^{\mathbb{Z}}$ is compact and so is every hull as above.

The minimality is an important property of the hull of (primitive) substitution sequences (see ahead), as well as the existence of a unique ergodic measure, and up to now rigorous and numerical studies have revealed just one dynamical behavior in each minimal component (usually rigorous results are restricted to generic or full measure sets). So, as a first step in the study of such new class of systems, in this work we address the problem of minimality of the hull of hybrid sequences in case their respective parent potentials are almost periodic. Setting a product metric, the dynamics, with respect to which one considers minimality, is on the product space of the hulls of the two parent potentials, and is generated by the (natural) product shift

$$\sigma(u, v)_n = (u_{n+1}, v_{n+1}). \tag{3}$$

We use the same notation for the shift in two and one-dimensional sequences. In order to investigate the minimality of the product spaces it turns out to be important hybridizing not only of v and u, but also of elements of their one-dimensional orbits; namely, to consider $\mathcal{I}_\kappa(v, \sigma^j(u))$, for each $j \in \mathbb{Z}$.

A finite word w is indexed $a_0 a_1 \ldots a_{|w|-1}$, $a_i \in \mathcal{A}$, where $|w|$ denotes the length of w. Given a set of infinite words X, the *language* of X, $\mathcal{L}(X)$, is the set of finite words occurring in some $w \in X$.

Let us describe some substitution rules which generate sequences of interest for this work; details can be found in refs. [2, 18]. Given a finite alphabet $\mathcal{A}$ a substitution is a map $\xi : \mathcal{A} \to \mathcal{A}^*$. Its iterations are defined by concatenation, that is, $\xi(abc) := \xi(a)\xi(b)\xi(c)$, $\xi^{n+1}(a) := \xi(\xi^n(a))$, $n \geq 1$. A substitution is *primitive* if there exists $k \in \mathbb{N}$ so that for every $a \in \mathcal{A}$ the word $\xi^k(a)$ contains all letters of $\mathcal{A}$. All substitutions in this work are primitive (see ref. [16] for some nonprimitive substitutions as potentials of Schrödinger operators).

A fixed point of a substitution is a sequence $u \in \mathcal{A}^{\mathbb{N}}$ such that $\xi(u) = u$. In order to exist, it must be the case that u_0 is the first letter of $\xi(u_0)$. It is known that if ξ is primitive, there is some l such that ξ^l has a fixed point,[(18)] so it is no loss to assume ξ has a fixed point.

Fibonacci (Fcc), Period Doubling (PD) and Thue-Morse (TM) substitution sequences are constructed with an alphabet of two letters $\{a, b\}$ through the rules

$$a \mapsto ab,\ b \mapsto a \quad \text{(Fcc)}, \qquad a \mapsto ab,\ b \mapsto ba \quad \text{(TM)},$$

$$a \mapsto ab,\ b \mapsto aa \text{ (PD)}.$$

Beginning with a (the seed) and applying successively the substitution rules (the growing rules), aperiodic sequences are obtained; e.g., the Thue-Morse sequence is given by

$$abbabaabbaababba\cdots$$

The paper folding (PF) sequence can be obtained with an alphabet of four letters $\{1, 2, 3, 4\}$, the substitution

$$1 \mapsto 12, \qquad 2 \mapsto 32, \qquad 3 \mapsto 14, \qquad 4 \mapsto 34,$$

(the seed is 1) and then applying the literal map $1, 2 \mapsto a$ and $3, 4 \mapsto b$.

We then use these substitution sequences to define our potentials V; we take $V_n = -1$ if the n-th letter of the sequence is a and $V_n = 1$ in case it is b. There are standard ways of extending such substitution potentials for negative values of n.[5,14] We do not have to deal with this issue in numerical simulations because we take a large finite sample of N sites, using the initial wavefunction concentrated on position $N/2$, i.e., $\psi_n(t = 0) = \delta_{N/2,n}$, $n \geq 0$. This is the procedure we use to construct almost periodic substitution potentials V.

It is known that the spectrum of the operator (1) with finite-valued aperiodic and almost periodic potentials has no absolutely continuous component (primitive substitutions are included)[14,15]; although from a rigorous point of view the lack/presence of eigenvalues in cases of primitive substitution sequences is an open question, as already remarked, no strong evidence of the presence of eigenvalues and localization was found yet.

Given a substitution ξ over a finite alphabet $\mathcal{A}$, denote by M_ξ its substitution matrix, i.e., $M_\xi = a_{w,w'}$, where $a_{w,w'}$ is the number of occurrences of the letter w' in $\xi(w)$. ξ is a *Pisot substitution* if the dominant eigenvalue of M_ξ has modulus greater than one, while all the other eigenvalues have absolute values strictly less than one. For example, the matrix substitution for TM and Fcc substitution are

$$M_{\mathrm{TM}} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad M_{\mathrm{Fcc}} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

whose dominant eigenvalues are 2 and $(1 + \sqrt{5})/2$, respectively. The dominant eigenvalue of the PD substitution is 2, but it is not Pisot, since the other eigenvalue is -1. PF is not Pisot either.

3. MINIMALITY OF HYBRID HULLS

Let v and u denote almost periodic sequences and $\Omega(v)$, $\Omega(u)$ be their respective hulls. In the product space $\Omega(v) \times \Omega(u)$ we have the shift defined by $\sigma(x, y) = (\sigma(x), \sigma(y))$. This dynamics does not imply that the product space is minimal if the parent hulls $\Omega(v)$ and $\Omega(u)$ are minimal. The orbit of a point (x, y) is $\mathcal{O}(x, y) = \{\sigma^n(x, y) : n \in \mathbb{Z}\}$. For each κ, there is a correspondence between elements of this product space and hybrid sequences $\mathcal{I}_\kappa(v_l, u_l)$, $v_l \in \Omega(v)$, $u_l \in \Omega(u)$. The potential is a real-valued function on one such sequence. It is of interest to know whether the potential is almost-periodic. We present in this section some results concerning minimality on the product space of minimal subsets of $\mathcal{A}^{\mathbb{Z}}$.

Given $\epsilon > 0$, the subset of integer numbers

$$\{n \in \mathbb{Z} : \ d(\sigma^n(x), x) < \epsilon\}$$

is called the set of ϵ-periods of $x \in X \subset \mathcal{A}^{\mathbb{Z}}$. When X is minimal, the above set is *syndetic*, i.e., there is an integer m such that any interval $[n, n+m] \subset \mathbb{Z}$ intersects it. Recall that x is almost periodic iff that set is syndetic for all $\epsilon > 0$; in this case of finite-valued sequences this is equivalent to the fact that every finite word in x appears with bounded gaps.

There is an alternative view of periods in terms of words, or equivalently the cylinder sets generated by them. If $a \in \mathcal{A}^{\mathbb{Z}}$, let $R_a(w)$ denote the set of integers n such that

$$a_n a_{n+1} \dots a_{n+|w|-1} = w.$$

Thus $R_a(w)$ is the set of integers n for which w is a *prefix* of $\sigma^n a$. It can be ordered

$$R_a(w) = \{\alpha_i, i \in \mathbb{Z} : \alpha_i < \alpha_{i+1}\}$$

for some arbitrary choice of α_0. The minimality of X is equivalent to the fact that for each finite word w that occurs in X there is an integer $m(w)$ so that $\alpha_{k+1} - \alpha_k < m(w), \forall k$ (i.e., w occurs with bounded gaps). Similarly, for $b \in Y \subset \mathcal{A}^{\mathbb{Z}}$, Y minimal, let $R_b(u) = \{\beta_j, \ \ j \in \mathbb{Z} : \beta_j < \beta_{j+1}\}$ (the notation should be clear).

In the product space $X \times Y$ we seek a description of the possible minimal sets under the shift and metric

$$D((a, b), (c, d)) := d_X(a, c) + d_Y(b, d), \quad a, c \in X, \quad b, d \in Y.$$

The existence of these minimal sets is a consequence of $X \times Y$ compactness and Zorn's Lemma.

Picture $X \times Y$ as the orbit closure of the union of $(a, \sigma^n(b))$, $n \in \mathbb{Z}$. If we represent the sequence a along a horizontal lattice $(\cdot, 0) \subset \mathbb{Z}^2$ and $\sigma^r(b)$ along vertical lattices, each attached to the corresponding horizontal position $(r, 0)$, the orbit $(a, \sigma^r b)$ is the left translation of the horizontal line $(\cdot, 0)$. Analogously the orbit of $(\sigma^k(a), b)$ may be followed by pulling horizontally the line at $(\cdot, k)$.

We begin to address the question about minimality of $\overline{\mathcal{O}(a, b)}$ by asking if, as one sits at different positions along the horizontal axis, one sees the same pair of finite words u, w upwards and to the right respectively, infinitely often. While this certainly happens at each β_n and α_n alternatively upwards and to the right, one is interested in these words appearing at the same time *and* with bounded gaps.

Proposition 1. *If $X \subset \mathcal{A}^{\mathbb{Z}}$ and $Y \subset \mathcal{B}^{\mathbb{Z}}$ are minimal sets, then $X \times Y$ decomposes into finitely many minimal sets.*

Proof: Suppose on the contrary that we had infinitely many invariant sets M_i. Choose a point in each M_i and an open set U_i containing it but with no intersection with M_j, $j > i$. Complete this cover with $V_i = M_i \setminus U_i$ (recall that U_i is also closed since we are dealing with product of cylinders). From the cover of U_i and V_i's we cannot extract a finite subcover, but $X \times Y$ is compact. □

Theorem 1.17 in ref. [11] yields a point whose orbit closure is a minimal set. We can show that this holds for every point in $X \times Y$, whenever X and Y are minimal sets.

Proposition 2. *Suppose $X, Y \subset \mathcal{A}^{\mathbb{Z}}$ are minimal. Given a point $z \in X \times Y$, its orbit closure $\overline{\mathcal{O}(z)}$ is minimal.*

Proof: Pick a point $(a, \sigma^j b)$ from a minimal set $M \subset X \times Y$. Then, for any $\epsilon > 0$, there exists a sequence $(n_p)_{p\in\mathbb{N}}$, $n_p \nearrow \infty$, $|n_{p+1} - n_p|$ bounded such that

$$D((a, \sigma^j(b)), \sigma^{n_p}(a, \sigma^j(b))) < \epsilon.$$

Now pick a point $(\sigma^k a, \sigma^l b) \in X \times Y$. For any $n_p > h = \max\{|l - j|, |k|\}$

$$\begin{aligned} D(\sigma^k(a, \sigma^{l-k}(b)), \sigma^{n_p+k}(a, \sigma^{l-k}(b))) &= d_X(\sigma^k a, \sigma^{n_p+k} a) \\ &\quad + d_Y(\sigma^l b, \sigma^{n_p+l} b) \le 2^k d_X(a, \sigma^{n_p} a) \\ &\quad + 2^{|l-j|} d_Y(\sigma^j b, \sigma^{n_p+j} b) < 2^{h+1}\epsilon \end{aligned}$$

and this can be made arbitrarily small. Since any $z \in X \times Y$ belongs to the closure of the orbit of some $(\sigma^k a, \sigma^l b)$, the proposition is proved. □

This proves the assertion in the abstract

Corollary 1. *If $X, Y \subset \mathcal{A}^{\mathbb{Z}}$ are minimal sets, then a sequence $z \in X \times Y$ is almost periodic, as well as any sequence obtained from it by some real-valued function defined on $X \times Y$.*

In what follows, unless stated on the contrary, we assume that X and Y are minimal sets. Now the question is to characterize when the product $X \times Y$ is minimal.

Proposition 3. *Suppose there exists a sequence $n_k \nearrow \infty$ so that $\sigma^{n_k} a \to a^*$ and $\{\sigma^{n_k+l} b : n_k\}$, for some l fixed, is dense in Y. Then $X \times Y$ is minimal.*

Proof: Our hypothesis asserts that (a^*, Y) is contained in the orbit closure $\overline{\mathcal{O}(a, \sigma^l(b))}$. Due to Proposition 2, it is enough to show that there is a dense orbit in $X \times Y$. Given $(x, y) \in X \times Y$ and $\epsilon > 0$

$$D((\sigma^n a, \sigma^{n+l} b), (x, y)) \leq D((\sigma^n a, \sigma^{n+l}(b)), (\sigma^j(a^*), y)) + D((\sigma^j(a^*), y), (x, y)).$$

Since X is minimal, the second term may be made less than $\epsilon/3$, and this fixes j. Now

$$D((\sigma^n a, \sigma^{n+l}(b)), (\sigma^j(a^*), y)) = d(\sigma^n(a), \sigma^j(a^*)) + d(\sigma^{n+l}(b), y).$$

We note that along the subsequence $n = n_k + j$ the first term is less than $\epsilon/3$ for every n_k sufficiently large. The set of points $\sigma^{n_k+l} b$, $n_k > N$, is dense in Y, N a fixed arbitrary integer. Therefore, if $z = \sigma^{-j} y$, there exists n_k so that $d(\sigma^{n_k+l} b, z) < \epsilon'$. Choosing ϵ' small enough yields $d(\sigma^{n_k+(j+l)} b, y) < \epsilon/3$, for some $n_k > N$. □

Therefore, if $X \times Y$ is not minimal, then for every convergent sequence $\sigma^{n_k}(a) \to a^*$ one has that $\sigma^{n_k+l}(b)$ is not dense, for any l.

We know that the dynamics of a map on the torus $\mathbb{T}^d$: $T(\theta) = \theta + \alpha \pmod 1$ is ergodic when α is rationally independent. By coding this dynamics with a partition along each circle, we get a symbolic sequence which is semi-conjugate to the original dynamics.[1] This is an example of a product of two sequences spaces which is minimal. We say that (w, u) is a *prefix* of (a, b) if w is a prefix of a and u is a prefix of b. It is easy to characterize lack of minimality in terms of the language of $X \times Y$. Indeed, if $\mathcal{L}(a, b)$ denote the set of words of $(a, b) \in X \times Y$, we have $\mathcal{L}(a, b) \subset \mathcal{L}(a) \times \mathcal{L}(b)$. Hence, if $X \times Y$ is not minimal, for each invariant set $M \subset X \times Y$, there are words $r \in \mathcal{L}(a)$ and $s \in \mathcal{L}(b)$ such that (r, s) does not occur in M.

Remark 1. If a is an almost periodic sequence, note that $\mathcal{L}(a) = \mathcal{L}(\Omega(a))$.

By hull of a substitution we understand the hull of any of its fixed points. For primitive substitution sequences we get a simple criterion for minimality of the product of their hulls. Recall that this case is our choice of prototypes of hybrid quasicrystals. The argument comes from the proof of a result Hansel in ref. [13] related to Cobham's Theorem (see also refs. [7, 10]). Recall that two positive numbers θ and ϑ are multiplicatively independent if the equation $\theta^l = \vartheta^k$ holds only for $l = k = 0$.

Theorem 1. *Let ξ and ζ be two primitive substitutions on the (finite) alphabets $\mathcal{A}$ and $\mathcal{B}$, respectively, and denote by X and Y their respective hulls under the shift. If M_ξ and M_ζ have multiplicatively independent dominant eigenvalues, then $X \times Y$ is minimal.*

Proof: Let $a = \xi(a) = (a_j)_{j\in\mathbb{Z}}$ and $b = \zeta(b) = (b_j)_{j\in\mathbb{Z}}$ be fixed points of the corresponding substitutions and $M = \overline{\mathcal{O}(a, b)}$. If $X \times Y$ is not minimal, then $M \neq X \times Y$ and there is a finite word (r, s) in $X \times Y$ that does not occur in M. Thus, for any r_0, r_1, s_0, s_1, $|r_0| = |s_0|$, $(r_0 r r_1, s_0 s s_1)$ does not occur in the orbit of (a, b) either.

Since the substitutions are primitive, there is a k so that for all $n \geq k$ the words $\xi^n(w)$, $w \in \mathcal{A}$, contain r, and $\zeta^n(u)$, $u \in \mathcal{B}$, contain s. We choose r_0 and r_1 so that $\xi^n(w) = r_0 r r_1$ above. Then s_0 and s_1 are chosen so that $\zeta^n(u)$ contains $s_0 s s_1$. We conclude that for any pair $(w, u) \in \mathcal{A} \times \mathcal{B}$ there is some n_0 (which may be taken *big*) so that

$$(\xi^{n_0}(w), \zeta^{n_0}(u)) \text{ is not a prefix of } \sigma^k(a, \sigma^{-l} b) \text{ for every } k \text{ and some } l. \tag{4}$$

Let θ and ϑ be the dominant eigenvalues of M_ξ and M_ζ respectively. Consider the subsets of $\mathbb{N}$

$$E(X) = \{|\xi(a_0 a_1 \ldots a_m)|, m > 0\},$$
$$E(Y) = \{|\zeta(b_0 b_1 \ldots b_m)|, m > 0\}.$$

$E(X)$ contains some of the positions where the words $\xi^j(w)$, $\forall j, \forall w \in \mathcal{A}$, occur in a. By Lemma 2 in ref. [13], for large enough m, these positions are the integer numbers closest to $a\theta^{pj} + b$ for some integer $p > 0$, and real $a > 0, b$. The same holds for $E(Y)$, in that it contains integers closest to $a'\vartheta^{qj} + b'$, for some integer $q > 0$, and real $a' > 0, b'$. But θ^p and ϑ^q are multiplicatively independent, so the set of ratios

$$\frac{a\theta^{pj} + b}{a'\vartheta^{qj} + b'}$$

is dense in $\mathbb{R}^+$. Therefore, it must be the case that the intersections $E(X) \cap E(Y)$ and $E(X) \cap \{E(Y) - l\}$, for any l, are not empty. This contradicts (4). □

Remark 2. In the case of Pisot substitutions, the proof is simpler in that the $E(X)$ will contain integer numbers close to θ^{pj}, while $E(Y)$ contains integers close to, ϑ^{qj}, for integer $j > 0$. If θ and ϑ are multiplicatively independent, the same argument follows.

Corollary 2. *If* Ω_{TM}, Ω_{Fcc} *and* Ω_{PD} *are the hulls of the indicated substitution sequences, then the products* $\Omega_{\mathrm{TM}} \times \Omega_{\mathrm{Fcc}}$ *and* $\Omega_{\mathrm{PD}} \times \Omega_{\mathrm{Fcc}}$ *are minimal.*

One can investigate whether some product spaces generated by constant length substitutions are not minimal in a case by case analysis. For instance if ξ denotes the Thue-Morse substitution and η is the period-doubling substitution, then contained in the above described $X \times Y$, one has the following subset Ω. Let

$\mathcal{B}$ denote the four letter alphabet: $\{(a, a), (a, b), (b, a), (b, b)\}$. On $\mathcal{B}$ we define the substitution

$$\zeta(x, y) := (\xi(x), \eta(y)).$$

Explicitly, $\zeta(a, a) = (ab, ab) = (a, a)(b, b)$, $\zeta(a, b) = (ab, aa) = (a, a)(b, a)$, $\zeta(b, a) = (ba, ab) = (b, a)(a, b)$ and $\zeta(b, b) = (ba, aa) = (b, a)(a, a)$. This substitution ζ can be shown to be primitive with two fixed points. The fixed points belong to the same hull, since the languages of the fixed points of a primitive substitution coincide. In a four letter alphabet, Berstel has considered a substitution isomorphic to ζ when constructing square-free words.(3) Let $u = abbabaab\cdots$ be one fixed point of the Thue-Morse substitution and $w = abaaabab\cdots$ be the aperiodic fixed point of the period-doubling substitution. We can see that Ω is an invariant minimal subset strictly contained in $X \times Y$ by noticing that, while $(abba, baaa)$ is a prefix of $(u, \sigma w)$, it does not occur in any point of the orbit $\sigma^j(u, w)$.

Similarly, we have analyzed the substitutions defined in an eight letter alphabet by the product of period doubling and Rudin-Shapiro, and the product of period doubling and paper-folding. These substitutions are *semi-primitive*, in the sense of ref. [5], see also ref. [17] where semi-primitiveness is shown for Rudin-Shapiro substitution. There is a sub-alphabet, with six letters, in which they are primitive. These substitutions also have two fixed points. The same argument on the location of the letter b in the period doubling substitution leads to more than one invariant set in $X \times Y$.

4. NUMERICS OF THE MOMENT

In this section we report some numerical simulations of the moment $m_2(T)$ as a function of time T for some hybrid quasicrystals. Basis sets were usually of size 2^{14}, and the time evolution was done by integrating the Schrödinger equation using a sympletic integrator, as described in ref. [6]. The emphasis will be on hybrid substitution quasicrystals. It is expected that different minimal sets present different behavior of $m_2(T)$ and, with respect to numerics, this is the working setting accepted here.

In these numerical experiments we have mostly fixed $\kappa = 1/2$, but exceptions are explicitly mentioned. We also set $\lambda = 1$ (preliminary results indicate that the qualitative behavior is independent of $\lambda \neq 0$). The guide to the simulations was based on two properties used in Theorem 1, that is, the multiplicatively independent dominant eigenvalues of their substitution matrices.

First consider the hybridizing of TM and Fcc. The results are summarized in Figure 1. Different elements of the product of the hulls are obtained by keeping one sequence fixed and shifting the other before the combination. Although both sequences individually generate transport (for TM see the dashed line in Figure 2),

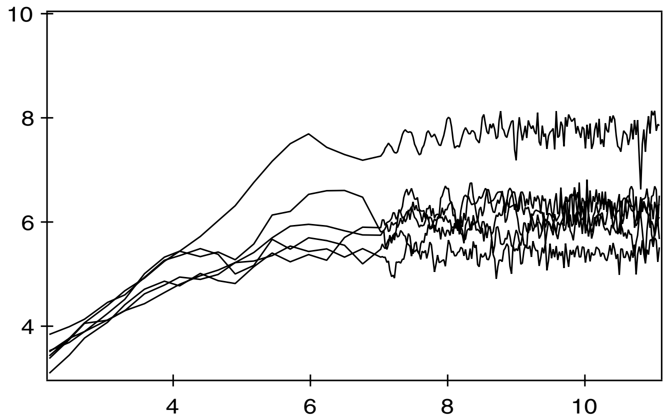

Fig. 1. The moment as function of time (log − log scale) for the combination of TM and FCC substitution sequences. The sequence FCC was kept fixed, while TM was shifted by 0, 1, ..., 5 in order to explore different elements of the product of their hulls.

when combined we have got only one behavior, in accordance with Theorem 1 and Corollary 2, since the hybrid hull is minimal in this case. This gives an example of numerical dynamical localization in an almost periodic sequence. As a complement to such simulations we have also considered $\kappa = 0.2$ and 0.8, and localization was always found; again the minimality seems to be the important property.

Another possibility we have investigated is when the two involved substitutions have multiplicative dependent dominant eigenvalues. The extreme case is for

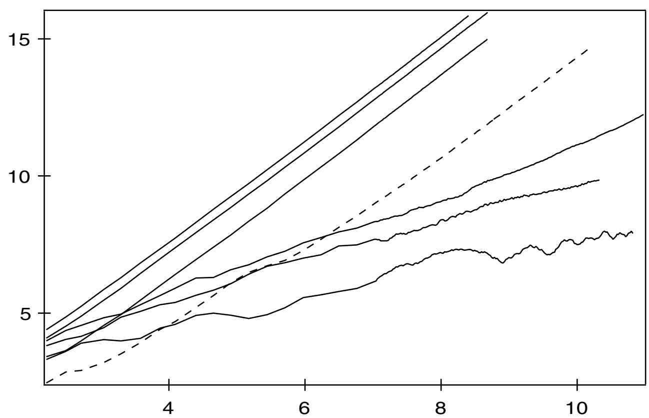

Fig. 2. The moment as function of time (log − log scale) for the combination of TM with itself shifted. The dashed line is for the original TM (no shift at all).

equal eigenvalues and we have selected this situation by hybridizing a substitution with shifts of itself. Figure 2 presents the results of these simulations for TM sequence; transport was found in all cases, although with different exponents β, indicating the presence of more than one minimal component in the product of the hulls; so there are different hybrid quasicrystals in this case. In Figure 2 the dashed line is for the original TM sequence. We have noted three distinct behaviors, with the dashed line as a border between them: for some shift values the moment follows the dashed line ($\beta \approx \beta_{\mathrm{TM}} = 1.8$), others present a range of $0 < \beta < \beta_{\mathrm{TM}}$ values, while others with near ballistic behavior (i.e., $\beta > 1.9$); that is, if $\beta > \beta_{\mathrm{TM}}$ then it is near the maximum possible value. We add that for the combination of Fcc with itself similar results were obtained (not shown), that is, transport prevails and different exponent values of β were found; however, without a case near the ballistic motion.

The same procedure was applied to the PD substitution. If no shift is applied to the sequences, then the original sequence is obtained and it cannot be considered a hybrid quasicrystal, although it is embedded in the product space. Except for this case, where $\beta_{\mathrm{PD}} \approx 1.78$, all other simulations clearly indicate a motion near the ballistic one (no figure is shown). It appears that the self-product of period-doubling substitution contains only two minimal components. We have also combined almost periodic substitution sequences with periodic ones (with periods up to 32), and quite distinct behaviors were found. A periodic sequence is also almost periodic and its hull has finitely many elements (as many as its period). We have hybridized PD, TM, PF and Fcc with periodic sequences and, depending on the choice of the period, for some cases we have found transport, with different values of β, but in some other periods we got localization. Figure 3 shows some

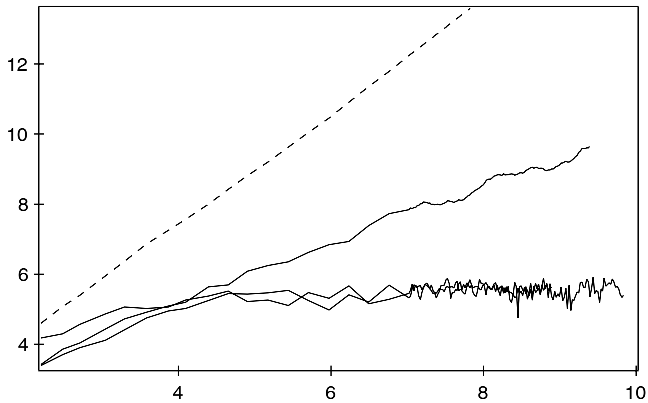

Fig. 3. The moment as function of time (log – log scale) for the combination of PF with periodic sequences. The periods were 4 (dashed), 16 (line), 7 and 10 (localization).

instances of PF combined with periodic sequences. Such long range perturbations have shown a rich range of possibilities.

5. CONCLUSIONS

In this work we considered hybrid quasicrystals, defined by the convex combination two parent finitely valued almost periodic sequences, as new models of one-dimensional quasicrystals. Hybridization of substitution sequences was given special attention.

We investigated in some generality the minimality of product spaces $X \times Y$, when both X and Y are minimal, and Section 3 presented a sufficient condition for primitive substitutions, which is the multiplicative independence of the eigenvalues of their substitution matrices. Minimality is well known when the metric on the sequence space is given by the sup-norm,[19] but requires extra work in the setting of finitely valued sequences.

Some hybrid potentials were inserted into Schrödinger equation and the time evolution of concentrated initial conditions numerically investigated; the interest was in localization and transport in such structures. In order to classify our numerical results we have adopted the pragmatic position that elements in the same minimal set should generate similar time evolutions. This was confirmed in cases our analytical results proved minimality for the product of minimal hulls, and suggested the presence of more than one minimal component in other cases. The figures presented in Section 4 illustrate these behaviors. The hybridization with periodic sequences was also numerically considered.

The numerical results suggest a rigorous investigation of localization in some hybrid quasicrystals. This could be accomplished by proving that the Lyapounov exponent γ in these sequences is uniformly positive, that is, the existence of $c > 0$ such that $\gamma > c > 0$.

To our knowledge, this result would be relevant since minimal sequences generated by primitive substitutions have been shown to have zero Lyapounov exponent by the following reasoning. Recall that $\mathcal{L}(\Omega)$ denotes the language of the minimal subshift Ω, and let $[v]$ be the cylinder set defined by the word v:

$$[v] \equiv \{\omega \in \Omega : \omega_1 \dots \omega_{|v|} = v\}$$

Let ν be a σ-invariant probability on (Ω, σ) and $n \in \mathbb{N}$ and $\mathcal{L}_n(\Omega)$ the set of words of length n occurring in Ω. Define

$$\eta_\nu(n) = \min\{\nu([w]) : w \in L_n(\Omega)\}$$

Boshernitzan's condition, first considered in subshifts related to interval exchange transformations,[4] may be written as

$$\limsup_{n\to\infty} n\eta_\nu(n) > 0.$$

It is proven that the family of ergodic operators $(H_\omega)_{\omega\in\Omega}$, when Ω is a minimal subshift satisfying Boshernitzan's condition, have zero Lyapounov exponent everywhere in their spectrum, which is a Cantor set of zero Lebesgue measure.[9]

We can see that Boshernitzan's condition does not hold in hybrid quasicrystals. For an almost periodic hybrid sequence $z \in \Omega$, the complexity $p_z(n)$, which counts the number of words of length n in z, is at least of the order of n^2, because the complexity of each component in a hybrid sequence is at least of order n. On the other hand, the measure of any cylinder must be inversely proportional to the complexity, because

$$p_z(n) \min_{v\in\mathcal{L}_n(\Omega)} \nu([v]) < \sum_{v\in\mathcal{L}_n(\Omega)} \nu([v]) = 1.$$

for any probability ν.

This leaves one important theoretical question, to pursue the possibility of Anderson localization in minimal hybrid quasicrystals.

REFERENCES

1. P. Alessandri, *Codages de rotations et basses complexités*, Ph.D. Thesis, University Aix-Marseille 2 (1996).
2. F. Axel and D. Gratias (eds.), *Beyond Quasicrystals*, Les Editions de Physique, Springer-Verlag, Berlin (1995).
3. J. Berstel, Sur la construction de mots sans carrè, *Sem. Th. des Nombres de Bordeaux* **8** (1979–1980).
4. M. Boshernitzan, A condition for minimal interval exchange maps to be uniquely ergodic. *Duke Math. J.* **52**:723–752 (1985).
5. A. Bovier and J.-M. Ghez, Spectral properties of one-dimensional Schrödinger operators with potentials generated by substitutions. *Commun. Math. Phys.* **158**: 45–66 (1993). Erratum: *Commun. Math. Phys.* **166**:431–432 (1994).
6. T. O. Carvalho and C. R. de Oliveira, Spectra and transport in almost periodic dimers. *J. Stat. Phys.* **107**:1015–1030 (2002).
7. A. Cobham, On the base dependence of sets of numbers recognizable by finite automata. *Math. Syst. Th.* **6**:164–192 (1972).
8. D. Damanik, Strictly ergodic subshifts and associated operators, to appear in Simon Festschrift, *Proceedings of Symposia in Pure Mathematics*, American Mathematical Society, Providence. mp-arc 05-309.
9. D. Damanik and D. Lenz, A condition of Boshernitzan and uniform convergence in the multiplicative ergodic theorem. *Duke Math. J.* **133**:95–123 (2006).
10. F. Durand, A Generalization of Cobham's Theorem. *Th. Comp. Syst.* **31**:169–185 (1998).
11. H. Furstenberg, *Recurrence in Ergodic Theory and Combinatorial Number Theory*, Princeton Univ. Press, Princeton (1981).

12. F. Germinet and S. De Bièvre, Dynamical localization for discrete and continuous random Schrödinger operators. *Commun. Math. Phys.* **194**:323–341 (1998).
13. G. Hansel, Systèmes de numération indépendants et syndéticité, *Th. Comp. Science* **204**:119–130 (1998).
14. A. Hof, O. Knill and B. Simon, Singular Continuous Spectrum for Palindromic Schrödinger Operators. *Commun. Math. Phys.* **174**:149–159 (1995).
15. S. Kotani, Jacobi matrices with random potential taking finitely many values. *Rev. Math. Phys.* **1**:129–133 (1989).
16. M. V. Lima and C. R. de Oliveira, Uniform Cantor Singular Continuous Spectrum for Nonprimitive Schrödinger Operators. *J. Stat. Phys.* **112**:357–374 (2003).
17. Q.-H. Liu, B. Tan, Z.-X. Wen, and J. Wu, Measure zero spectrum of a class of Schrödinger operators. *J. Stat. Phys.* **106**:681–691 (2002).
18. M. Queffélec, *Substitution Dynamical Systems–Spectral Analysis*, Lect. N. Math. **1294**, Springer-Verlag, Berlin (1987).
19. K. Petersen, *Ergodic Theory*, Cambridge Univ. Press, Cambridge (1983).
20. B. Simon and T. Wolf, Singular Continuous Spectrum under rank one Perturbations and Localization for Random Hamiltonians. *Commun. Pure Appl. Math.* **39**:75–90 (1986).
21. L. Vuillon, Combinatoire des motifs d'une suite sturmienne bidimensionnelle. *Theor. Comp. Sc.* **209**:261–285 (1998).